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On the $\overline{\partial}$-problem and integrable equations

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Abstract

Using the $\overline{\partial}$-problem and dual $\overline{\partial}$-problem, we derive bilinear relations which allow us to construct integrable hierarchies in different parametrizations, their Darboux–Bäcklund transformations and to analyse constraints for them in a very simple way. Scalar KP, BKP and CKP hierarchies are considered as examples.

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There are different methods of constructing integrable equations and analysing their properties (see, e.g., [1–4]). The $\overline{\partial}$-dressing method proposed in [5] is, perhaps, one of the most effective of them. Recently, it has been applied successfully to several important problems in soliton theory (see, e.g., [6–11]).

In this paper we would like to draw the reader's attention to one more profitable aspect of the $\overline{\partial}$-dressing method. Namely, starting with the $\overline{\partial}$-problem and the dual $\overline{\partial}$-problem, we derive two important bilinear relations for the so-called Cauchy–Baker–Akhiezer (CBA) functions associated with different kernels R of the $\overline{\partial}$-problem. These relations provide us with simple variational relations for CBA functions and the $\overline{\partial}$-kernel R. In a simple unified manner, they generate integrable hierarchies in different parametrizations and corresponding bilinear Hirota identities. These bilinear relations are also convenient for the analysis of different constraints. It is shown how scalar BKP and CKP hierarchies arise within such an approach. We also demonstrate that a pole-type parametrization of evolutions leads to the continuous analogues of the Darboux system.

The $\overline{\partial}$-dressing method is based on the non-local $\overline{\partial}$-problem for a function with some normalization (see, e.g., [5–7]). We start with the following pair of $\overline{\partial}$-problems dual to each other:

$$\frac{\partial \chi'(\lambda,\mu)}{\partial\overline{\lambda}} = \pi\,\delta(\lambda-\mu) + \iint_{\mathbb{C}} \mathrm{d}\nu\wedge\mathrm{d}\overline{\nu}\,\chi'(\nu,\mu)\,R'(\nu,\lambda) \tag{1}$$

and

$$\frac{\partial \chi^*(\lambda,\rho)}{\partial\overline{\lambda}} = -\pi\,\delta(\lambda-\rho) - \iint_{\mathbb{C}} \mathrm{d}\nu\wedge\mathrm{d}\overline{\nu}\,R(\lambda,\nu)\,\chi^*(\nu,\rho) \tag{2}$$

where $\lambda \in \mathbb{C}$, a bar denotes complex conjugation and $\delta(\lambda)$ is the Dirac delta-function. The functions χ, χ^*, R and R^* depend both on λ and $\bar{\lambda}$, μ and $\bar{\mu}$, etc. To simplify the notation we will omit the dependence on $\bar{\lambda}, \bar{\mu}, \bar{\rho}, \bar{\nu}$, etc. At $\lambda \to \mu$ we have

$$\chi'(\lambda, \mu) = \frac{1}{\lambda - \mu} + \chi_r'(\lambda, \mu) \qquad \chi^*(\lambda, \mu) = -\frac{1}{\lambda - \mu} + \chi_r^*(\lambda, \mu)$$

where χ_r' and χ_r^* are regular functions. Solutions of the $\bar{\partial}$-problem with such properties have been introduced in different contexts in [6, 12]. We shall refer to $\chi(\lambda, \mu)$ as the CBA functions. Furthermore, we assume that $R'(\nu, \lambda) = R(\nu, \lambda) = 0$ for $\nu \in G$, $\lambda \in G$ where G is a certain domain in $\mathbb{C}$ and $\mu, \rho \in G$. So the functions $\chi_r(\lambda, \mu)$ and $\chi_r^*(\lambda, \mu)$ are analytic in G with respect to both variables. Typically, $G = D_0$ or $G = D_0 \cup D_\infty$ where D_0 and D_∞ are the unit discs around the origin $\lambda = 0$ and around the infinity $\lambda = \infty$, respectively. In general, χ and R in (1) and (2) are matrix-valued functions.

To derive desired bilinear relations we first multiply from the right both sides of equation (1) by $f_1(\lambda)\, \chi^*(\lambda, \rho)$ and then multiply both sides of equation (2) by $\chi'(\lambda, \mu)\, f_2(\lambda)$ from the left where $f_1(\lambda)$ and $f_2(\lambda)$ are arbitrary matrix-valued functions. Summing up the obtained equations, one obtains

$$\begin{aligned} \frac{\partial \chi'(\lambda, \mu)}{\partial \bar{\lambda}}\, f_1(\lambda)\, \chi^*(\lambda, \rho) + \chi'(\lambda, \mu)\, f_2(\lambda)\, \frac{\partial \chi^*(\lambda, \rho)}{\partial \bar{\lambda}} &= \pi\, \delta(\lambda - \mu)\, f_1(\lambda)\, \chi^*(\lambda, \rho) - \pi\, \delta(\lambda - \rho)\, \chi'(\lambda, \mu)\, f_2(\lambda) \\ &\quad + \iint_{\mathbb{C}} \mathrm{d}\nu \wedge \mathrm{d}\bar{\nu} \big[\chi'(\nu, \mu)\, R'(\nu, \lambda)\, f_1(\lambda)\, \chi^*(\lambda, \rho) \\ &\quad - \chi'(\lambda, \mu)\, f_2(\lambda)\, R(\lambda, \nu)\, \chi^*(\nu, \rho)\big]. \end{aligned} \tag{3}$$

Integrating (3) with respect to λ over $\mathbb{C}$, one obtains

$$\begin{aligned} \iint_{\mathbb{C}} \mathrm{d}\lambda \wedge \mathrm{d}\bar{\lambda} \left[\frac{\partial \chi'(\lambda, \mu)}{\partial \bar{\lambda}}\, f_1(\lambda)\, \chi^*(\lambda, \rho) + \chi'(\lambda, \mu)\, f_2(\lambda)\, \frac{\partial \chi^*(\lambda, \rho)}{\partial \bar{\lambda}}\right] &= 2\pi \mathrm{i}\, [\chi'(\rho, \mu)\, f_2(\rho) - f_1(\mu)\, \chi^*(\mu, \rho)] \\ &\quad + \iint_{\mathbb{C}} \mathrm{d}\lambda \wedge \mathrm{d}\bar{\lambda} \iint_{\mathbb{C}} \mathrm{d}\nu \wedge \mathrm{d}\bar{\nu}\, \chi'(\nu, \mu) \\ &\quad \times \big[R'(\nu, \lambda)\, f_1(\lambda) - f_2(\nu)\, R(\nu, \lambda)\big]\, \chi^*(\lambda, \rho). \end{aligned} \tag{4}$$

Then integration of (3) over $\mathbb{C}/G$ gives

$$\begin{aligned} \iint_{\mathbb{C}/G} \mathrm{d}\lambda \wedge \mathrm{d}\bar{\lambda} \left[\frac{\partial \chi'(\lambda, \mu)}{\partial \bar{\lambda}}\, f_1(\lambda)\, \chi^*(\lambda, \rho) + \chi'(\lambda, \mu)\, f_2(\lambda)\, \frac{\partial \chi^*(\lambda, \rho)}{\partial \bar{\lambda}}\right] &= \iint_{\mathbb{C}/G} \mathrm{d}\lambda \wedge \mathrm{d}\bar{\lambda} \iint_{\mathbb{C}/G} \mathrm{d}\nu \wedge \mathrm{d}\bar{\nu}\, \chi'(\lambda, \mu) \\ &\quad \times \big[R'(\nu, \lambda)\, f_1(\lambda) - f_2(\nu)\, R(\nu, \lambda)\big]\, \chi^*(\lambda, \rho). \end{aligned} \tag{5}$$

Considering equation (5) with $R' = R$ (hence, $\chi' = \chi$) and $f_1 = f_2 = 1$, one readily obtains the well known result $\chi^*(\mu, \rho) = \chi(\rho, \mu)$ [8, 9].

The bilinear identities (4) and (5) (with $\chi^*(\lambda, \rho) = \chi(\rho, \lambda)$) are the fundamental bilinear relations within the $\bar{\partial}$-dressing method. We shall show that these relations provide us with integrable hierarchies and the basic formulae associated with them in a simple and transparent way.

In what follows we will consider the particular case of $f_1(\lambda) = f_2(\lambda) = f(\lambda)$ and $\frac{\partial f(\lambda)}{\partial \bar{\lambda}} = 0$ at $\lambda \in \mathbb{C}/G$ and assume that $f(\lambda)$ and $\chi(\lambda, \mu)$ have no discontinuities on ∂G.

Thus, our starting bilinear relations are

$$2\pi\mathrm{i}\,[f(\mu)\,\chi(\rho,\mu)-\chi'(\rho,\mu)\,f(\rho)] = -\iint_{\mathbb{C}} \mathrm{d}\lambda\wedge\mathrm{d}\overline{\lambda}\,\chi'(\lambda,\mu)\,\frac{\partial f(\lambda)}{\partial\overline{\lambda}}\,\chi(\rho,\lambda)$$
$$-\iint_{\mathbb{C}} \mathrm{d}\lambda\wedge\mathrm{d}\overline{\lambda}\iint_{\mathbb{C}} \mathrm{d}\nu\wedge\mathrm{d}\overline{\nu}\,\chi'(\nu,\mu)\big[R'(\nu,\lambda)\,f(\lambda)-f(\nu)\,R(\nu,\lambda)\big]\chi(\rho,\lambda) \tag{6}$$

$$\int_{\partial G}\mathrm{d}\lambda\,\chi'(\lambda,\mu)\,f(\lambda)\,\chi(\rho,\lambda) = \iint_{\mathbb{C}/G}\mathrm{d}\lambda\wedge\mathrm{d}\overline{\lambda}\iint_{\mathbb{C}/G}\mathrm{d}\nu\wedge\mathrm{d}\overline{\nu}\,\chi'(\nu,\mu)$$
$$\times\big[R'(\nu,\lambda)\,f(\lambda)-f(\nu)\,R(\nu,\lambda)\big]\,\chi(\rho,\lambda). \tag{7}$$

At $f=1$ relation (7) gives

$$\chi'(\rho,\mu)-\chi(\rho,\mu) = -\frac{1}{2\pi\mathrm{i}}\iint_{\mathbb{C}/G}\mathrm{d}\lambda\wedge\mathrm{d}\overline{\lambda}\iint_{\mathbb{C}/G}\mathrm{d}\nu\wedge\mathrm{d}\overline{\nu}\,\chi'(\nu,\mu)$$
$$\times\big[R'(\nu,\lambda)-R(\nu,\lambda)\big]\,\chi(\rho,\lambda). \tag{8}$$

Thus, in particular,

$$\frac{\delta\chi(\rho,\mu)}{\delta R(\nu,\lambda)} = -\frac{1}{2\pi\mathrm{i}}\,\chi(\rho,\lambda)\,\chi(\nu,\mu) \qquad \rho,\mu\in G \quad \nu,\lambda\in\mathbb{C}/G. \tag{9}$$

Then in the case of general degenerate variation of R formula (8) provides us with an explicit transformation of χ. Indeed, let

$$R'(\nu,\lambda) = R(\nu,\lambda)-2\pi\mathrm{i}\sum_{k=1}^{n}A_k(\nu)\,B_k(\lambda) \tag{10}$$

where A_k and B_k are arbitrary functions. Substituting (9) into (8), one obtains

$$\chi'(\rho,\mu)-\chi(\rho,\mu) = \sum_{k=1}^{n}X_k^{*'}(\mu)\,X_k(\rho) \tag{11}$$

where

$$\begin{aligned} X_k^{*'}(\mu) &= \iint_{\mathbb{C}/G}\mathrm{d}\nu\wedge\mathrm{d}\overline{\nu}\,\chi'(\nu,\mu)\,A_k(\nu) \\ X_k(\rho) &= \iint_{\mathbb{C}/G}\mathrm{d}\lambda\wedge\mathrm{d}\overline{\lambda}\,B_k(\lambda)\,\chi(\rho,\lambda). \end{aligned} \tag{12}$$

It follows from (11) that

$$X_i^{*'}(\mu)-X_i(\mu) = \sum_{k=1}^{n}X_k^{*'}(\mu)\,C_{k\,i} \tag{13}$$

where

$$C_{k\,i} = \iint_{\mathbb{C}/G}\mathrm{d}\lambda\wedge\mathrm{d}\overline{\lambda}\iint_{\mathbb{C}/G}\mathrm{d}\nu\wedge\mathrm{d}\overline{\nu}\,B_k(\nu)\,\chi(\lambda,\nu)\,A_i(\lambda). \tag{14}$$

Using (13) and (11), one obtains

$$\chi'(\rho,\mu) = \chi(\rho,\mu)+\sum_{i,k=1}^{n}X_i(\mu)\,\big[(1-C)^{-1}\big]_{i\,k}\,X_k(\rho) \tag{15}$$

where $X_i(\lambda)$ are given by (12). This formula describes dressing of the CBA function $\chi(\lambda, \mu)$ under a generic degenerate transformation (10) of the $\bar{\partial}$-kernel on an arbitrary background $R(\nu, \lambda)$. In the particular case of a degenerate background kernel $R(\nu, \lambda)$ and within a different approach, similar formula have been derived recently in [13].

Now let us consider continuous transformations. The simplest of them are given by a similarity transformation of the kernel R

$$R'(\nu, \lambda) = G(\nu)\, R(\nu, \lambda)\, G^{-1}(\lambda) \tag{16}$$

where $G(\lambda)$ is a matrix-valued function. We assume that $G(\lambda)$ is analytic in $\mathbb{C}/G$ and continuous on ∂G. Considering formulae (6) and (7) with $f(\lambda) = G(\lambda)$, we conclude that under the transformations (16), the following bilinear relations hold:

$$\chi'(\rho, \mu)\, G(\rho) - G(\mu)\, \chi(\rho, \mu) = -\frac{1}{2\pi \mathrm{i}} \iint_G \mathrm{d}\lambda \wedge \mathrm{d}\bar{\lambda}\, \chi'(\lambda, \mu)\, \frac{\partial G(\lambda)}{\partial \bar{\lambda}}\, \chi(\rho, \lambda) \tag{17}$$

and

$$\int_{\partial G} \mathrm{d}\lambda\, \chi'(\lambda, \mu)\, G(\lambda)\, \chi(\rho, \lambda) = 0. \tag{18}$$

It is easy to check that these two relations are equivalent to each other.

Representing $G(\lambda)$ as $G(\lambda) = g'(\lambda)\, g^{-1}(\lambda)$ and denoting $\chi(\lambda, \mu) \equiv \chi(\lambda, \mu; g)$, $\chi'(\lambda, \mu) \equiv \chi(\lambda, \mu; g')$, one rewrites (18) in the form

$$\int_{\partial G} \mathrm{d}\lambda\, \chi'(\lambda, \mu; g')\, g'(\lambda)\, g^{-1}(\lambda)\, \chi(\rho, \lambda; g) = 0 \tag{19}$$

that is the generalized Hirota bilinear identity introduced and discussed in [14, 15]. In the particular case $\mu = \rho = 0$ it represents the celebrated Hirota bilinear identity (see, e.g., [3]). It was shown in [15] that the identity (19) provides an effective tool to describe and analyse the so-called generalized integrable hierarchies and hierarchies of corresponding singularity manifold equations.

Formulae (17) and (18) define finite continuous transformations. For infinitesimal transformations $G(\lambda) = 1 + \varepsilon\, \omega(\lambda)$, $\delta R(\lambda, \mu) = \varepsilon \frac{\partial R(\lambda, \mu)}{\partial \tau}$ and $\delta \chi(\lambda, \mu) = \varepsilon \frac{\partial \chi(\lambda, \mu)}{\partial \tau}$ where $\varepsilon \to 0$ and τ is the transformation parameter . The infinitesimal version of the formulae (16)–(18) looks like

$$\frac{\partial}{\partial \tau} R(\nu, \lambda) = \omega(\nu)\, R(\nu, \lambda) - R(\nu, \lambda)\, \omega(\lambda) \tag{20}$$

$$\frac{\partial}{\partial \tau} \chi(\rho, \mu) = \omega(\mu)\, \chi(\rho, \mu) - \chi(\rho, \mu)\, \omega(\rho) - \frac{1}{2\pi \mathrm{i}} \iint_G \mathrm{d}\lambda \wedge \mathrm{d}\bar{\lambda}\, \chi(\lambda, \mu)\, \frac{\partial \omega(\lambda)}{\partial \bar{\lambda}}\, \chi(\rho, \lambda) \tag{21}$$

$$\frac{\partial}{\partial \tau} \chi(\rho, \mu) = \frac{1}{2\pi \mathrm{i}} \int_{\partial G} \mathrm{d}\lambda\, \chi(\lambda, \mu)\, \omega(\lambda)\, \chi(\rho, \lambda). \tag{22}$$

Formula (21) and (22) are equivalent to each other, but in some cases one of them is more convenient than the other. Formula (22) with $\varepsilon\, \omega(\lambda) = \delta g(\lambda)\, g^{-1}(\lambda)$ can also be found in [15], while a version of formula (21) with integration over $\mathbb{C}$ has been derived in [9] (see also [7]). A formula similar to (22) has been derived in [12] by a different method.

Equations (21) and (22) define integrable deformations of the CBA function since the $\bar{\partial}$-problems (1) and (2) allow us to construct wide classes of exact solutions for them. The concrete form of these integrable evolutions is defined by a form of the function $\omega(\lambda)$. In the rest of the paper we will consider only scalar equations. With the simplest choice $\omega(\lambda) = \frac{1}{2\pi \mathrm{i}} \frac{1}{\lambda - a}$ where $a \in G$ is a parameter, one obtains ($\tau = a$) for $a \neq \rho$, $a \neq \mu$

$$\frac{\partial \chi(\rho,\mu)}{\partial a} = \left(\frac{1}{\mu - a} - \frac{1}{\rho - \mu}\right) \chi(\rho,\mu) + \chi(a,\mu)\,\chi(\rho,a) \qquad \rho \neq \mu. \tag{23}$$

In terms of the function $\beta(\rho,\mu)$ defined as $\beta(\rho,\mu,a) = -\frac{\rho\,(\mu - a)}{\mu\,(\rho - a)}\,\chi(\rho,\mu,a)$ equation (23) looks like

$$\frac{\partial \beta(\rho,\mu)}{\partial a} = \beta(a,\mu)\,\beta(\rho,a). \tag{24}$$

Equation (23) (or (24)) describes integrable deformations of the CBA function due to the motion of position a of the pole of $\omega(\lambda)$ (see also [8]). In addition to this analytic meaning, it has a pure geometric interpretation. Namely, equation (24) together with its cyclic permutations is nothing but the continuous analogue of the Darboux system $\frac{\partial \beta_{ik}}{\partial X_l} = \beta_{il}\,\beta_{lk}$ which describes the triply conjugate system of surfaces in $\mathbb{R}^3$ [16]. This old geometric system and its discrete generalizations have attracted considerably interest recently (see, e.g., [7, 8, 11, 13]). Note that in our approach the continuous Darboux system (24) arises in a scalar case. In a different context such a fact has already been mentioned in [15, 17].

The continuous Darboux system (24) possesses all properties of the standard Darboux system. In particular, the functions X_i and X_i^* defined by the formula (12) represent the tangent vectors, while the function

$$\phi = \iint_{\mathbb{C}/G} \mathrm{d}\lambda \wedge \mathrm{d}\bar{\lambda} \iint_{\mathbb{C}/G} \mathrm{d}\mu \wedge \mathrm{d}\bar{\mu}\, A(\mu)\,\chi(\lambda,\mu)\,B(\lambda) \tag{25}$$

is a position vector. Formula (15) gives an explicit transformation of solution of the continuous Darboux system (24). It has a form of the standard Darboux–Lévy transformation (see, e.g., [18]). The choice $\omega(\lambda) = \frac{1}{2\pi \mathrm{i}} \sum_{k=1}^{n} \frac{1}{\lambda - a_k}$ in (21) and (22) leads to the system of n separated continuous Darboux systems.

If now we parametrize the function g in (19) as $g(\lambda) = \exp\left(\sum_{n=1}^{\infty} t^n/\lambda^n\right)$ then we have infinite set of infinitesimal shifts of variables t_n with $\omega_n = g_{t_n}\, g^{-1} = \frac{1}{\lambda^n}$ and the corresponding equations (22) take the form

$$\begin{aligned}\frac{\partial \chi(\rho,\mu)}{\partial t_n} &= \left(\frac{1}{\mu^n} - \frac{1}{\rho^n}\right) \chi(\rho,\mu) + \frac{1}{(n-1)!}\left\{\frac{\partial^{n-1}}{\partial \lambda^{n-1}}\,[\chi(\lambda,\mu)\,\chi(\rho,\lambda)]\right\}_{\lambda=0} \\ &n = 1, 2, 3, \ldots.\end{aligned} \tag{26}$$

This hierarchy of equations is equivalent to that studied in [15] and hence the hierarchy (26) describes the generalized Kadomtsev–Petviashvili (KP) hierarchy which include the KP hierarchy itself, the modified KP hierarchy and the hierarchy of KP singularity manifold equations.

It is known that the times t_n and the pole-type parametrizations of the KP hierarchy are connected by the Miwa transformation $t_n = \frac{1}{n} \sum_{i=1}^{\infty} a_i^n$ [19]. In fact, due to the relation $\frac{\partial}{\partial a} = \sum_{n=1}^{\infty} a^{n-1} \frac{\partial}{\partial t_n}$, the equivalence of the infinite hierarchy (26) and equation (23) is an easy check (see also [17]).

The special choice of the function $\omega(\lambda)$ may provide interesting deformations. For example, let us put $\omega(\lambda) = S(\lambda)$, where $S(\lambda)$ is the Schwarz function of the curve ∂G. The Schwarz function completely characterizes the curve and $\bar{\lambda} = S(\lambda)$ at $\lambda \in \partial G$ [20]. Thus for boundaries ∂G such that $S(\lambda)$ is analytic outside G, one has the deformations

$$\frac{\partial}{\partial \tau}\,\chi(\rho,\mu) = -\int_{\partial G} \mathrm{d}\lambda\,\bar{\lambda}\,\chi(\lambda,\mu)\,\chi(\rho,\lambda) = -\int_{\partial G} \mathrm{d}\lambda\,\chi(\lambda,\mu)\,S(\lambda)\,\chi(\rho,\lambda). \tag{27}$$

Such deformations are defined by the form of the boundary ∂G of the domain G. If G is the unit disc D_0 then $S(\lambda) = \frac{1}{\lambda}$ [20] and the deformation (27) is of KP type (26). In the case when G is a circle of the radius 1 with the centre at λ_0, then $S(\lambda) = \frac{1}{\lambda - \lambda_0} + \overline{\lambda}_0$ and the deformation (27) ($\tau = \lambda_0$) coincides with (23).

Not only continuous integrable equations but also discrete ones can be easily derived from the basic bilinear equations (6) and (7). For instance, treating the transformation (16) with $G(\lambda) = \frac{1}{\lambda - a}$ as the shift in the discrete variable n, namely, $R'(\nu, \lambda; n) = R(\nu, \lambda; n+1) = T_a\, R(\nu, \lambda; n)$ one readily obtains from (7) the equation

$$(T_a - 1)\ \psi(\rho, \mu) = T_a\, \psi(a, \mu) \cdot \psi(\rho, a) \qquad \rho \neq \mu \tag{28}$$

where

$$\psi(\rho, \mu; n) = (\mu - \rho) \left(\frac{\mu - a}{\rho - a}\right)^n \chi(\rho, \mu; n) \tag{29}$$

which is the discrete analogue of the Darboux system (24). A discrete Darboux system has been derived in [8] and then has been studied intensively over the last few years in the context of discrete integrable nets (see, e.g., [11]).

The basic bilinear relations (6) and (7) are also useful for studying the constraints of generic integrable hierarchies. Here we will show how the scalar BKP and CKP hierarchies [3] arise within this approach. For this purpose it is sufficient to use relations (6) and (7) with $R'(\nu, \lambda) = R(-\lambda, -\nu)$ and assume that the kernel R satisfies the constraint

$$R(-\lambda, -\nu)\, F(\lambda) = F(\nu)\, R(\nu, \lambda) \tag{30}$$

where $F(\lambda)$ is a function obeying the condition $F(-\lambda) = \pm F(\lambda)$. In this case the domain G has to be symmetric under the change $\lambda \to -\lambda$. Such types of constraints in matrix case have been discussed recently in [9, 11].

First, we note that a solution of the $\overline{\partial}$-problem (1) with the kernel $R'(\nu, \lambda) = R(-\lambda, -\nu)$ is given by $\chi'(\nu, \lambda) = \chi(-\lambda, -\nu)$. Then, the relation (7) with $f(\lambda) = F(\lambda)$ and the kernel R which satisfies (30) takes the form

$$\int_{\partial G} \mathrm{d}\lambda\, \chi(-\mu, -\lambda)\, F(\lambda)\, \chi(\rho, \lambda) = 0. \tag{31}$$

As in the generic case, we have the generalized Hirota identity (19) but now the transformations (16) have to be compatible with the constraint (30). This implies that $g^{-1}(\lambda) = g(-\lambda)$. Due to the constraint (31) the identity (19) (with $g^{-1}(\lambda) = g(-\lambda)$) can be rewritten in an equivalent form.

First we consider the case $F = 1$. So, $R(-\lambda, -\nu) = R(\nu, \lambda)$. Then the constraint (31) implies that $\chi(-\mu, -\rho) = \chi(\rho, \mu)$. Hence the generalized Hirota identity (19) looks like ($G = D_0$)

$$\int_{\partial D_0} \mathrm{d}\lambda\, \chi(\lambda, \mu; g')\, g'(\lambda)\, g(-\lambda)\, \chi(-\lambda, -\rho; g) = 0. \tag{32}$$

At $\mu = \rho = 0$, and with the parametrization of g by standard KP times ($g(\lambda) = \exp\left[\sum_{n=1}^{\infty} t_{2n-1}/\lambda^{2n-1}\right]$), relation (32) coincides with the Hirota bilinear identity for a scalar CKP hierarchy.

The treatment of the constraint (30) with $F = \frac{1}{\lambda}$ is a little bit more involved. First, the constraint (31) gives

$$\frac{1}{\mu}\, \chi(\rho, -\mu) + \frac{1}{\rho}\, \chi(\mu, -\rho) = \chi(\rho, 0)\, \chi(\mu, 0). \tag{33}$$

Then the identity (19) with $g(\lambda) = \exp\left[\sum_{n=1}^{\infty} t_{2n-1}/\lambda^{2n-1}\right]$ and $t_1' = t_1 + \varepsilon$, $\varepsilon \to 0$ implies

$$\int_{\partial D_0} \mathrm{d}\lambda \left[\left(\frac{\partial}{\partial t_1} + \frac{1}{\lambda}\right) \chi(\lambda, -\mu)\right] \cdot \chi(\rho, \lambda) = 0. \tag{34}$$

Subtracting equation (31) with $F = \frac{1}{\lambda}$ from (34), one obtains

$$\int_{\partial D_0} \mathrm{d}\lambda \left[\left(\frac{\partial}{\partial t_1} + \frac{1}{\lambda}\right) \chi(\lambda, -\mu) - \frac{1}{\lambda}\, \chi(\mu, -\lambda)\right] \cdot \chi(\rho, \lambda) = 0. \tag{35}$$

For $\mu = 0$ the quantity in the bracket in (35) has no singularities in D_0. Hence, equation (35) implies that $\left(\frac{\partial}{\partial t_1} + \frac{1}{\rho}\right) \chi(\rho, 0) = \frac{1}{\rho}\, \chi(0, -\rho)$ or equivalently

$$g^{-1}(\lambda)\, \chi(0, \lambda) = -\lambda \frac{\partial}{\partial t_1}\, [g(-\lambda)\, \chi(-\lambda, 0)]. \tag{36}$$

With the use of (36) one rewrites the Hirota identity (19) with $\mu = \rho = 0$ in the form

$$\frac{\partial}{\partial t_1} \int_{\partial D_0} \mathrm{d}\lambda\, \lambda\, \chi(\lambda, 0; g')\, g'(\lambda)\, g(-\lambda)\, \chi(-\lambda, 0; g) = 0$$

and finally as

$$\int_{\partial D_0} \frac{\lambda\, \mathrm{d}\lambda}{2\pi \mathrm{i}}\, \chi(\lambda, 0; g')\, g'(\lambda)\, g(-\lambda)\, \chi(-\lambda, 0; g) = -1. \tag{37}$$

This relation is just the Hirota bilinear identity for the scalar BKP hierarchy (see [3]) written in terms of wavefunctions with the normalization $\frac{1}{\lambda}$ as $\lambda \to 0$. In terms of times t_{2n-1} the equations of the BKP hierarchy are given by equations (26) with $n = 2k - 1$, $k = 1, 2, 3, \ldots$ and $\mu = 0$ or $\rho = 0$. It is a straightforward check that the constraint (33) is compatible with these equations.

In similar manner one can treat multicomponent KP hierarchies, the Toda lattice hierarchy and other types of constraints.

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